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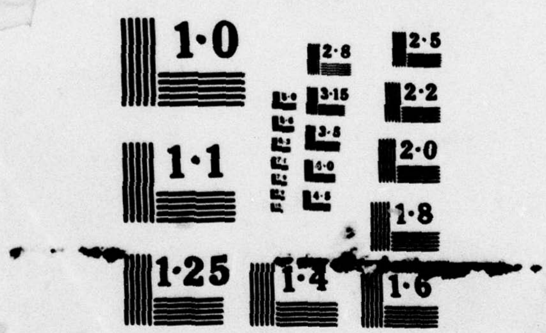

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RESEARCH ON THE DIRECT AND INVERSE PROBLEMS OF RADIATIVE AND NEUTRON TRANSPORT IN PLANE-STRATIFIED AND SPHERICALLY SYMMETRIC MEDIA

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TABLE OF CONTENTS

		Page
1.0	INTRODUCTION	1
2.0	RESULTS OF RESEARCH	2
3.0	PAPERS FOR PUBLICATION	3

APPENDIX A COPIES OF PUBLICATIONS IN 3.0

1.0 INTRODUCTION

In the contract proposal the Principal Investigators set out objectives for their research which involved finding new techniques for solving problems in radiative transfer and neutron transport. We also proposed to start an investigation of the inverse problem which has only been taken up in recent years. The direct problem, which is the usual problem treated, attempts to solve the transport equations when the phase function (which characterizes the emission and absorption properties of the atoms or molecules in the medium) is known. By contrast, the inverse problem is concerned with finding the phase function when sufficient information is given about the gross transmission and absorption properties of the medium as a whole. As indicated in the proposal, our initial approach has been directed to one-dimensional problems, i.e. problems in which stratification occurs along one of the axes, say the z-axis. We are happy to report that we have made considerable progress in our objectives.

2.0 RESULTS OF RESEARCH

The results of our research is rather fully documented in three papers, two of which have been published and the third of which is in the process of publication. We believe that our achievements are considerable, considering that they were completed within one year. The inverse problem, particularly, has broken new ground and offers the possibility of characterizing atoms or molecules of a medium when the transmittance or albedo are known. Moreover, it also suggests the possibility of a rational method of designing nuclear piles corresponding to neutron flux requirements.

3.0 PAPERS FOR PUBLICATION

As mentioned above, the accomplishments are given in three papers, two of which are published and the third of which has been accepted for publication. Two reprints corresponding to the published papers and a preprint of the paper to be published are given as part of this final report. In the present section we give only the abstracts.

(a) "A Variational Principle for Transport Theory," Journal of Mathematical Physics, 19, 1258 (1978).

ABSTRACT

A maximal variational principle is used to construct an infinite medium Green's function for treating the boundary value problems of the linear transport theory (neutron and radiative). For the neutrons we consider the one-speed case and correspondingly for the radiative transfer the monochromatic case. The scattering properties of the medum are presumed to be dependent on the relaxation length. Thus, for the neutrons the secondary production function depends on the neutron's relaxation length and for the radiative transfer the albedo for single scattering is dependent on the optical depth. These two functions are kept arbitrary so that a large class of problems can be covered. The basic principle involves a functional which is an absolute maximum when the trial function is an exact solution of an integral equation of the Fredholm type. The kernel of the integral equation is required to satisfy certain symmetry and boundlessness properties. We also exhibit an interesting relation between the absolute maximal and Schwinger's stationary variational principles, which in general is neither a maximum nor a minimum.

(b) "Direct-Inverse Problems in Transport Theory. 1. The Inverse Problem," Journal of Mathematical Physics, 19, 1793 (1978).

ABSTRACT

For the inverse problem treated here we use the results of an experiment which measures the total angular-dependent column density as compared to the measurement which provides information on the angle integrated spatial-dependent angular density (or the specific intensity). We use two methods of approach. One, the Legendre expansion method and two a maximal variational principle. In particular we demonstrate how the variational principle yields a very convenient representation of the scattering kernel (or the phase function) in terms of a basis consisting of Case eigenfunctions for the isotropically scattering medium.

(c) "Direct-Inverse Problems in Transport Theory. The Inverse Albedo Problem for a Finite Medium," Journal of Mathematical Physics (to be published).

ABSTRACT

In this paper we deal with the inverse problem in radiative transfer, which is equally applicable to the neutron transport, for a finite homogeneous medium. We give a method for computing the scattering function and the albedo for single scattering. The solution is given in terms of Legendre expansion of the scattering function. A decomposition of the equation of transfer is also given in which the relation between the direct and the inverse problem is exhibited via the principles of invariance. A relation of this work with Case's method of approach is outlined. This work should be of practical value to the problems associated with remote sensing of the terrestrial atmosphere and the neutron piles.

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Direct-Inverse Problems in Transport Theory.
1. The Inverse Problem

A Variational Principle for Transport Theory

Direct-Inverse Problems in Transport Theory,
The Inverse Albedo Problem for a Finite Medium

Direct-inverse problems in transport theory. 1. The inverse problem[a)]

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(Received 3 March 1978)

For the inverse problem treated here we use the results of an experiment which measures the total angular-dependent column density as compared to the measurement which provides information on the angle integrated spatial-dependent angular density (or the specific intensity). We use two methods of approach. One, the Legendre expansion method and two a maximal variational principle. In particular we demonstrate how the variational principle yields a very convenient representation of the scattering kernel (or the phase function) in terms of a basis consisting of Case eigenfunctions for the isotropically scattering medium.

1. INTRODUCTION

The object of the direct problem of transport theory, (e.g., neutron or radiative) the usual problem treated, is to find the distribution function for a given scattering function. By contrast, the inverse problem involves construction of the scattering function from the results of some simple experiment which gives some knowledge of the distribution function. For the inverse problem in general, it is usually not clear with regard to the nature of the minimal set of measurements one needs in order to construct the scattering function uniquely. For instance, consider a relatively simple case of transport of monoenergetic neutrons in a medium with a homogeneous mixture of different nuclear species. If the angular scattering differential cross section associated with each species is different, then it would appear to be difficult to devise an experiment from which any set of measurements can lead to a unique conclusion for the shape of the angular differential cross section for each nuclear species. If one had a medium consisting of only one nuclear species, which is not necessarily homogeneously distributed, then certain relatively simple measurements will lead to a unique determination of the differential cross section. The reason is simply that in that case the secondary production function, which is the ratio of the rate at which the neutrons are elastically scattered to the rate at which the scattering involves all nuclear processes, become independent of the density of that nuclear species and depends only on its scattering properties. Similar considerations apply to problems of radiative transfer and, *inter alia*, the problem of transport of hyperthermal electrons in a plasma containing a substantial amount of neutral species of various kinds.

2. DIRECT INVERSE PROBLEMS

For our presentation of the direct inverse problems in transport theory we consider an infinite medium with a plane source at $x=x_0$ emitting one-speed neutrons in a direction whose cosine is μ_0. For the one-dimensional time-independent, and azimuthally symmetric case, the standard neutron transport equation is[1]

$$\mu\frac{\partial\Psi}{\partial x}(x,\mu\to\mu_0)+\Psi=\frac{c}{2}\int_{-1}^{1}d\mu'\,f(\mu'\to\mu)\Psi(x,\mu'\to\mu_0)+\delta(x-x_0)\delta(\mu-\mu_0) \tag{1}$$

where, $c<1$ (nonmultiplying medium), Ψ is the distribution function and $f(\mu'\to\mu)$ is the normalized symmetric scattering function, i.e.,

$$\tfrac{1}{2}\int_{-1}^{1}d\mu'\,f(\mu'\to\mu)=1. \tag{2}$$

For the direct problem, $f(\mu'\to\mu)$ is given and Ψ is to be determined everywhere. For the inverse problem certain results of an experiment are given and then $f(\mu'\to\mu)$ is to be constructed from those measurements.

In an earlier paper,[2] Case presents an elegant method of solving the inverse problem. The essence of his procedure is that, if one expands the scattering function in terms of Legendre polynomials so that

$$f(\mu'\to\mu)=\sum_{l=0}^{\infty}(2l+1)f_lP_l(\mu')P_l(\mu), \tag{3}$$

then for the unit isotropic plane source the expansion coefficients

$$g(l)=1-cf_l \tag{4}$$

are determined from the measurement of the density of neutrons as a function of x. Therefore, as Case has shown from the direct problem, one has the spectral representation of the total density $[\Phi(x)]$

$$\Phi(x)=\int_0^{\infty}\frac{d\rho(\nu)}{\nu}e^{-|x|/\nu}, \tag{5}$$

where $\rho(\nu)$ is the spectral function given by

$$\frac{d\rho(\nu)}{\nu}=\frac{d\nu}{N(\nu)},\quad -1<\nu<1$$
$$=\sum_i\frac{\delta(\nu-\nu_i)d\nu}{N_i},\quad |\nu|>1, \tag{6}$$

where $N(\nu)$ and N_i are the normalizations of the eigenfunctions of the continuous and discrete spectra, respectively. For the inverse problem he states that, given the measurement of the total density $\Phi(x)$, one knows quite a bit about the spectral density from Eq. (5). We will not give further

[a)]Research sponsored by the Office of Naval Research.

details here as to how this argument actually leads to the determination of the expansion coefficients $g(l)$ and hence $f(\mu'\to\mu)$ (see Refs. 2 and 3 for a complete discussion).

We wish to present a different method in the present paper for dealing with the inverse problem. In contrast to the data used in Case's approach we require the measurement of the angular column density $M_0(\mu,\mu_0)$ of neutrons for a unit plane source emitting neutrons in one direction μ_0 (monodirectional). In other words, the experiment should provide the information on the quantity

$$M_0(\mu,\mu_0)=\int_{-\infty}^{\infty} dx\,\Psi(x,\mu\to\mu_0) \tag{7}$$

where $\Psi(x,\mu\to\mu_0)$ is the angular density of neutrons satisfying Eq. (1).

Clearly $M_0(\mu,\mu_0)$ is the angle-dependent zeroth moment of the distribution function (the angular density) which is related to the scattering function. That relation is readily obtained from Eq. (1) by integration with respect to x. With the appropriate boundary conditions that Ψ vanishes at $x=\pm\infty$, we have from Eq. (1)

$$M_0(\mu,\mu_0)=\delta(\mu-\mu_0)+\frac{c}{2}\int_{-1}^{1}d\mu'\,f(\mu'\to\mu)M_0(\mu',\mu_0). \tag{8}$$

This is a Fredholm type of an integral equation for $M_0(\mu,\mu_0)$ which involves the scattering function as the kernel. If we let

$$S_0(\mu,\mu_0)=\int_{-1}^{1}d\mu'\,f(\mu'\to\mu)M_0(\mu',\mu_0) \tag{9}$$

then $S_0(\mu,\mu_0)$ (the emission term) also satisfies a *similar* type of an integral equation. That equation is readily obtained by multiplying Eq. (8) by $f(\mu\to\mu_1)$ and integrating with respect to μ. With change of names of variables we have

$$S_0(\mu,\mu_0)=f(\mu_0\to\mu)+\frac{c}{2}\int_{-1}^{1}d\mu'\,f(\mu'\to\mu)S_0(\mu',\mu_0). \tag{10}$$

For convenience we write Eq. (8) as

$$M_0(\mu,\mu_0)=\delta(\mu-\mu_0)+\frac{c}{2}S_0(\mu,\mu_0). \tag{11}$$

Before addressing the Direct Inverse problems, we wish to point out that if in the integral equations (8) and (10) the kernel $f(\mu\to\mu_0)$ satisfies certain properties then there is a maximal variational principle[4] which can be used to solve such equations. Explicitly, if $f(\mu\to\mu_0)$ satisfies the following properties:

(a) $f(\mu\to\mu_0)$ is symmetric in μ,μ_0,

(b) $f(\mu\to\mu_0)>0$, for $-1<(\mu,\mu_0)<1$,

(c) $\frac{c}{2}\int_{-1}^{1}d\mu'\,f(\mu\to\mu_0)<1,\ -1<\mu_0<1,\ c<1,$

then the functional

$$F_S[n]=\int_{-1}^{1}d\mu\,n(\mu,\mu_0)\Big[2f(\mu,\mu_0)-n(\mu,\mu_0)+\frac{c}{2}\int_{-1}^{1}d\mu'\,f(\mu'\to\mu)n(\mu',\mu_0)\Big] \tag{12}$$

is an *absolute maximum* if and only if $n(\mu,\mu_0)$ is an exact solution of the integral equation (10). Or conversely, as in the inverse problem, if we consider Eq. (10) as an integral equation for $f(\mu_0\to\mu)$ so that

$$f(\mu_0\to\mu)=S_0(\mu,\mu_0)-\frac{c}{2}\int_{-1}^{1}d\mu'\,S_0(\mu',\mu_0)f(\mu'\to\mu), \tag{13a}$$

then the functional

$$F_f[n]=\int_{-1}^{1}d\mu\,n(\mu_0,\mu)\Big[2S_0(\mu,\mu_0)-n(\mu_0,\mu)-\frac{c}{2}\int_{-1}^{1}d\mu'\,S_0(\mu',\mu_0)n(\mu',\mu)\Big] \tag{13b}$$

is an absolute maximum if and only if $n(\mu_0,\mu)=f(\mu_0\to\mu)$ is the exact solution of the integral equation (13a). This result follows from the fact that $S_0(\mu,\mu_0)$ is symmetric and the operator Λ corresponding to the integral equation (13a), i.e.

$$\Lambda f=f \tag{13c}$$

is positive definite. In two previous papers by Kanal and Moses[5,6] such a variational principle was used to solve the problems of inverse scattering and also a demonstration of its application to the linear transport theory for the isotropic scattering inhomogeneous media was given. Here for the direct inverse problems we can proceed in two ways. One is to expand $f(\mu\to\mu_0)$ in terms of Legendre polynomials and use their orthogonality property to relate the expansion coefficients to the Legendre moments of $S_0(\mu,\mu_0)$. This method produces exact solutions. The other way is to use the variational principle. Use of either method will be dictated by the kind of an experiment that is required to relate the scattering function $f(\mu\to\mu_0)$ to the emission term $S_0(\mu,\mu_0)$ or vice versa. We shall illustrate both methods.

A. Inverse Problem by Legendre Expansion

First we note that if $f(\mu\to\mu_0)$ is symmetric then so is $S_0(\mu,\mu_0)$. Let us now expand $f(\mu\to\mu_0)$ in terms of Legendre polynomials so that

$$f(\mu\to\mu_0)=\sum_{n=0}^{\infty}(2n+1)f_nP_n(\mu)P_n(\mu_0). \tag{14}$$

Insert this expansion in Eq. (10) to obtain

$$S_0(\mu,\mu_0)=\sum_{n=0}^{\infty}(2n+1)f_nP_n(\mu)P_n(\mu_0).+\frac{c}{2}\sum_{n=0}^{\infty}(2n+1)f_nP_n(\mu)q_n(\mu_0), \tag{15}$$

where

$$q_n(\mu_0)=\int_{-1}^{1}d\mu\,P_n(\mu)S_0(\mu,\mu_0). \tag{16}$$

Using the orthogonality property of Legendre polynomials

$$\int_{-1}^{1}d\mu\,P_n(\mu)P_l(\mu)=\frac{2}{2n+1}\delta nl, \tag{17}$$

we conclude from Eq. (15) that

$$f_l=\frac{q_l(\mu_0)}{2P_l(\mu_0)+cq_l(\mu_0)}. \tag{18}$$

Note that f_l are independent of μ_0. In consequence one needs to make the measurement for any one direction μ_0 to determine all the expansion coefficients f_l. In particular, for $\mu_0=1$, i.e., for the source emitting neutrons parallel to the x axis, we have

$$f_l=\frac{q_l(1)}{2+cq_l(1)}. \tag{19}$$

This result is really not surprising for a homogeneous medium due to the fact that the medium is rotationally invariant. However, it is of prime importance for the construction of $f(\mu\to\mu_0)$. In particular, if the medium is not drastically anisotropically scattering, then one needs only a few expansion coefficients and this method becomes very useful.

Now the Legendre moments $q_l(\mu_0)$ of $S_0(\mu,\mu_0)$ can be determined from the measurement of the angular column density $M_0(\mu,\mu_0)$. In other words, given $M_0(\mu,\mu_0)$, we conclude from Eq. (11), that

$$q_l(\mu_0)=\frac{2}{c}\left[\left[\int_{-1}^{1} d\mu\, P_l(\mu)M_0(\mu,\mu_0)-P_l(\mu_0)\right]\right]. \tag{20}$$

The quantity c (the secondary production term) is obtained from Eq. (8) [or from Eq. (11)] by integration with respect to μ so that

$$c=1-\left[\left[\int_{-1}^{1} d\mu\, M_0(\mu,\mu_0)\right]\right]^{-1}. \tag{21}$$

This is a well-known result (cf. Ref. 1). However, from the inverse problem point of view we again note that c, being independent of μ_0, is determined from the experiment with the source emitting neutrons in any arbitrary direction.

It would be interesting to know the degree of anisotropy of the scattering medium. We can estimate that from the experiment if $q_l(\mu_0)$ were calculated for all μ_0, i.e., if we calculate all the Legendre moments of the column density [see for example Eq. (19)]. For in that case we obtain from Eq. (18) the relation

$$\int_{-1}^{1} d\mu\, P_n q_l(\mu)=\frac{4f_n}{(2n+1)(1-cf_n)}\,\delta nl. \tag{22}$$

In other words, $(P_n(\mu), q_l(\mu))$ form a biorthogonal set, but more importantly for $l=n$ we get

$$f_n=\frac{(2n+1)T_n}{4+c(2n+1)T_n}, \tag{23}$$

where

$$T_n=\int_{-1}^{1} d\mu\, P_n(\mu)q_n(\mu). \tag{24}$$

The f_n's will give a measure of the degree of anisotropy so that $f_n=0$, for $n>N$.

B. Inverse Problem by the Variational Principle

Expansion in terms of Legendre polynomials is useful when the situation is not too anisotropic. When one wishes to consider very anisotropic cases, the variational principle may be more useful. Thus, consider Eq. (13a) so that

$$f(\mu\to\mu_0)=S_0(\mu,\mu_0)-\frac{c}{2}\int_{-1}^{1} d\mu'\, S_0(\mu',\mu_0)f(\mu'\to\mu). \tag{25}$$

If $n(\mu,\mu_0)=f(\mu\to\mu_0)$ is the exact solution of Eq. (25), then the functional defined by (13b) is an absolute maximum. The value of the functional is

$$F_f[f]=\frac{2}{c}\,[S_0(\mu_0,\mu_0)-f(\mu_0\to\mu_0)] \tag{26a}$$

or

$$f_N(\mu_0\to\mu_0)\simeq S_0(\mu_0,\mu_0)-\frac{c}{2}F_{f_N}[f_N], \tag{26b}$$

where f_N is a class of trial functions. From (26a) we see that from the class of functions $\{f_N\}$, the function which maximizes the functional (13b) is the desired one. From (26b) one obtains an upper bound for $f(\mu_0\to\mu_0)$, i.e., forward scattering for any f_N used.[7]

To illustrate the application of the variational principle we now give an example. In analogy with the comparison potential technique in the inverse scattering theory,[8] we shall choose a sequence of trial functions for $f(\mu\to\mu_0)$ which is constructed from the complete set of eigenfunctions of the transport equation with *isotropic scattering*. In this way some very elegant results are obtained. Thus, we shall expand $f(\mu\to\mu_0)$ in terms of those eigenfunctions and find that the expansion coefficients satisfy decoupled integral equations analogous to Eq. (25) involving the same kernel $S_0(\mu,\mu_0)$.

For $f(\mu\to\mu_0)=1$, we have the completeness relation,

$$\delta(\mu-\mu_0)=\frac{1}{N_{0+}}[\mu\phi_{0+}(\mu)\phi_{0+}(\mu_0)-\mu\phi_{0-}(\mu)\phi_{0-}(\mu_0)]+\int_{-1}^{1}\frac{d\nu}{N(\nu)}\mu\phi_\nu(\mu)\phi_\nu(\mu_0), \tag{27}$$

where $\phi_{0\pm}(\mu)$, $\phi_\nu(\mu)$ are Case's[1] discrete and continuum eigenfunctions, respectively, defined by

$$\phi_{0\pm}(\mu)=\pm\frac{c\nu_0}{2}\frac{1}{\pm\nu_0-\mu}, \tag{28a}$$

$$\phi_\nu(\mu)=\frac{c\nu}{2}P\frac{1}{\nu-\mu}+\lambda(\nu)\delta(\nu-\mu), \tag{28b}$$

$$\lambda(\nu)=\tfrac{1}{2}[\Lambda^+(\nu)+\Lambda^-(\nu)], \tag{28c}$$

$$\Lambda(z)=1-\frac{cz}{2}\int_{-1}^{1}\frac{d\mu}{z-\mu}, \tag{28d}$$

$$\Lambda(\nu_0)=0, \tag{28e}$$

$$N_{0\pm}=\pm\frac{c\nu_0^3}{2}\left(\frac{c}{\nu_0^2-1}-\frac{1}{\nu_0^2}\right), \tag{28f}$$

$$N(\nu)=\Lambda^+(\nu)\Lambda^-(\nu). \tag{28g}$$

These eigenfunctions are orthogonal so that

$$\int_{-1}^{1} d\mu\, \phi_{0+}(\mu)\phi_{0-}(\mu)=0, \tag{29a}$$

$$\int_{-1}^{1} d\mu\mu\, \phi_{0\pm}^2(\mu)=N_{0\pm}, \tag{29b}$$

$$\int_{-1}^{1} d\mu\mu\, \phi_\nu(\mu)\phi_{\mu'}(\mu)=N(\mu)\delta(\nu-\nu'). \tag{29c}$$

We remark here that this set of eigenfunctions seems to be rather natural for the expansion of $f(\mu\to\mu_0)$ for the reason that, as we shall see later, for both extreme cases $f(\mu\to\mu_0)=1$ (isotropic) and $f(\mu\to\mu_0)=2\delta(\mu-\mu_0)$, the variational principle gives exact results. Now consider the expansion,

$$f(\mu\to\mu_0)=\frac{1}{N_{0+}}[\phi_{0+}(\mu_0)B_{0+}(\mu)-\phi_{0-}(\mu)B_{0-}(\mu)]+\int_{-1}^{1}\frac{d\nu}{N(\nu)}\phi_\nu(\mu_0)B_\nu(\mu) \tag{30}$$

so that

$$B_{0\pm}(\mu_0)=\int_{-1}^{1}d\mu\mu f(\mu\to\mu_0)\phi_{0\pm}(\mu) \tag{31a}$$

and

$$B_\nu(\mu)=\int_{-1}^{1}d\mu\mu f(\mu\to\mu_0)\phi_\nu(\mu). \tag{31b}$$

Since $f(\mu\to\mu_0)$ is symmetric, the expansion (30) on the right-hand side must also be symmetric in μ,μ_0. Inserting Eq. (30) in the integral equation (25) we get

$$\begin{aligned}&\frac{1}{N_{0+}}[\phi_{0+}(\mu)B_{0+}(\mu_0)-\phi_{0-}(\mu)B_{0-}(\mu_0)]+\int_{-1}^{1}\frac{d\nu}{N(\nu)}\phi_\nu(\mu)B_\nu(\mu_0)\\&=S_0(\mu,\mu_0)-\frac{c}{2}\int_{-1}^{1}d\mu'\,S_0(\mu',\mu_0)\\&\quad\times\left\{\frac{1}{N_{0+}}[\phi_{0+}(\mu)B_{0+}(\mu')-\phi_{0-}(\mu)B_{0-}(\mu')]+\int_{-1}^{1}\frac{d\nu}{N(\nu)}\phi_\nu(\mu)B_\nu(\mu')\right\}.\end{aligned} \tag{32}$$

Upon using the orthogonality properties (29a, b, c) in Eq. (32), we obtain

$$B_{0\pm}=A_{0\pm}(\mu)-\frac{c}{2}\int_{-1}^{1}d\mu'\,S_0(\mu',\mu)B_{0\pm}(\mu') \tag{33a}$$

and

$$B_\nu(\mu)=A_\nu(\mu)-\frac{c}{2}\int_{-1}^{1}d\mu'\,S_0(\mu',\mu)B_\nu(\mu'), \tag{33b}$$

where

$$A_{0\pm}(\mu)=\int_{-1}^{1}d\mu'\mu'\,\phi_{0\pm}(\mu')S_0(\mu',\mu) \tag{34a}$$

$$A_\nu(\mu)=\int_{-1}^{1}d\mu'\mu'\,\phi_\nu(\mu')S_0(\mu',\mu). \tag{34b}$$

By comparing integral equations (33a, b) with Eq. (25) we see the obvious parallel and note that variational principles analogous to that for Eq. (25) can be set up for $B_{0\pm}(\mu)$ and $B_\nu(\mu)$ to solve the inverse problem. As mentioned earlier the advantage of dealing with the integral equations for expansion coefficients $B_{0\pm}$ and B_ν is that in both extreme cases of isotropic scattering $[f(\mu\to\mu_0)=1]$ and when $f(\mu\to\mu_0)$ is purely forward peaked [monodirectional, i.e., $\frac{1}{2}f(\mu\to\mu_0)=\delta(\mu-\mu_0)$] the variational principle provides exact solutions for $f(\mu\to\mu_0)$. We may see that as follows: Consider Eq. (33a) for $B_{0\pm}(\mu_0)$ and let the trial function be

$$\Phi_{0\pm}(\mu)=\alpha_{0\pm}A_{0\pm}(\mu) \tag{35}$$

where $\alpha_{0\pm}$ is a discrete parameter. Also consider the functional for (33a) analogous to (13b)

$$F_{0\pm}[\Phi_{0\pm}]=\int_{-1}^{1}d\mu\,\Phi_{0\pm}(\mu)\left[2A_{0\pm}(\mu)-\Phi_{0\pm}(\mu)-\frac{c}{2}\int_{-1}^{1}d\mu'\,S_0(\mu',\mu)\Phi_{0\pm}(\mu')\right]. \tag{36}$$

Put $\Phi_{0\pm}(\mu)=\alpha_{0\pm}A_{0\pm}(\mu)$ in (36) to obtain

$$F_{0\pm}[A_{0\pm}]=\alpha_{0\pm}(q-\alpha_{0\pm})P_{0\pm}-\frac{c}{2}\alpha_{0\pm}^2Q_{0\pm}, \tag{37}$$

where

$$P_{0\pm}=\int_{-1}^{1}d\mu\,A_{0\pm}^2(\mu), \tag{38a}$$

$$Q_{0\pm}=\int_{-1}^{1}d\mu\,A_{0\pm}(\mu)\int_{-1}^{1}d\mu'\,S_0(\mu',\mu)A_{0\pm}(\mu'). \tag{38b}$$

Maximization of the functional $F_{0\pm}[A_{0\pm}]$, defined by Eq. (37), yields

$$\alpha_{0\pm}=\left(1+\frac{c}{2}\frac{Q_{0\pm}}{P_{0\pm}}\right)^{-1}. \tag{39}$$

Similarly for the continuum expansion coefficients $B_\nu(\mu)$ of Eq. (33b), if we consider the trial function

$$\Phi_\nu(\mu)=\alpha_\nu A_\nu(\mu), \tag{40}$$

then the maximization of the functional

$$F_\nu[\Phi_\nu]=\int_{-1}^{1}d\mu\,\Phi_\nu(\mu)\left[2A_\nu(\mu)-\Phi_\nu(\mu)-\frac{c}{2}\int_{-1}^{1}d\mu'\,S_0(\mu',\mu)\Phi_\nu(\mu')\right] \tag{41}$$

yields

$$\alpha_\nu=\left(1+\frac{c}{2}\frac{Q_\nu}{P_\nu}\right)^{-1}, \tag{42}$$

where

$$P_\nu=\int_{-1}^{1}d\mu\,A_\nu^2(\mu), \tag{43a}$$

$$Q_\nu=\int_{-1}^{1}d\mu\,A_\nu(\mu)\int_{-1}^{1}d\mu'\,S_0(\mu',\mu)A_\nu(\mu'). \tag{43b}$$

Hence, by replacing $B_{0\pm}$ by $\Phi_{0\pm}$ and B_ν by Φ_ν in Eq. (30), we have an approximate representation of the phase function, which is given by

$$\begin{aligned}f(\mu\to\mu_0)\cong\frac{1}{N_{0+}}&\left[\phi_{0+}(\mu_0)A_{0+}(\mu)\Big/\left(1+\frac{c}{2}\frac{Q_{0+}}{P_{0+}}\right)\right.\\&\left.-\phi_{0-}(\mu_0)A_{0-}(\mu)\Big/\left(1+\frac{c}{2}\frac{Q_{0-}}{P_{0-}}\right)\right]\\&+\int_{-1}^{1}\frac{d\nu}{N(\nu)}\phi_\nu(\mu_0)A_\nu(\mu)\Big/\left(1+\frac{c}{2}\frac{Q_\nu}{P_\nu}\right).\end{aligned} \tag{44}$$

We may now check the accuracy of the representation (44) for the two extreme cases when

$$f(\mu\to\mu_0)=\begin{cases}1 & \text{isotropic}\\ 2\delta(\mu-\mu_0) & \text{monodirectional.}\end{cases} \tag{45}$$

For the two cases we conclude from Eq. (25) that

$$S_0(\mu,\mu_0)=\begin{cases}\dfrac{1}{1-c} & \text{isotropic,}\\[2mm] \dfrac{2}{1-c}\,\delta(\mu-\mu_0) & \text{monodirectional.}\end{cases} \tag{46}$$

From Eqs. (38a, b) and Eqs. (43a, b) we obtain

$$\frac{Q_{0\pm}}{P_{0\pm}}=\frac{Q_\nu}{P_\nu}=\frac{2}{1-c} \tag{47}$$

for *both* isotropic and monodirectional phase functions, while

$$\left.\begin{aligned}A_{0\pm}&=\pm\nu_0\\ A_\nu(\mu)&=\nu\end{aligned}\right\}\quad\text{isotropic} \tag{48}$$

and

$$\left.\begin{aligned}A_{0\pm}(\mu)&=\frac{2}{(1-c)}\,\mu\phi_{0\pm}(\mu)\\ A_\nu(\mu)&=\frac{2}{1-c}\,\mu\phi_\nu(\mu)\end{aligned}\right\}\quad\text{monodirectional.} \tag{49}$$

Thus for the isotropic case we conclude from Eqs. (44) and (48) that

$$f(\mu\to\mu_0)=(1-c)\left(\frac{\nu_0}{N_{0+}}\left[\phi_{0+}(\mu_0)+\phi_{0-}(\mu_0)\right]+\int_{-1}^{1}\frac{d\nu\,\nu}{N(\nu)}\right). \tag{50}$$

But the right-hand side of Eq. (50) is merely a unity. This is easily seen by noting that Case's eigenfunctions satisfy the following relations:

$$\int_{-1}^{1}d\mu\mu\,\phi_\nu=\nu(1-c), \tag{51a}$$

$$\int_{-1}^{1}d\mu\mu\,\phi_{0\pm}(\mu)=(\pm\nu_0)(1-c). \tag{51b}$$

An integration of the completeness relation (27) with respect to μ gives

$$(1-c)\left(\frac{\nu_0}{N_{0+}}\,\phi_{0+}(\mu_0)+\phi_{0-}(\mu_0)+\int_{-1}^{1}\frac{d\nu\,\nu}{N(\nu)}\right)=1. \tag{52}$$

Hence for the isotropic case we obtain in Eq. (50), $f(\mu\to\mu_0)=1$ (as expected). For the monodirectional phase function, substitution of Eqs. (49) and (47) in Eq. (44) merely reproduces the completeness relation (27) save for a factor of two so that $f(\mu\to\mu_0)=2\delta(\mu-\mu_0)$; the factor of two should be there because of the normalization so that

$$\tfrac{1}{2}\int_{-1}^{1}d\mu\,f(\mu\to\mu_0)=1.$$

In conclusion we wish to point out that for the *direct problem* one can actually calculate the coefficients $A_{0\pm}(\mu)$ and $A_\nu(\mu)$ in the approximate eigenfunction representation (44) of $f(\mu\to\mu_0)$. Let us assume that the right-hand side of Eq. (44) is an exact representation for some function $f_0(\mu\to\mu_0)$ so that

$$f_0(\mu\to\mu_0)=\frac{1}{N_{0+}}\left(\phi_{0+}(\mu_0)A_{0+}(\mu)\alpha_{0+}-\phi_{0-}(\mu_0)A_{0-}(\mu)\alpha_{0-}\right)+\int_{-1}^{1}\frac{d\nu}{N(\nu)}\,\phi_\nu(\mu_0)A_\nu(\mu)\alpha_\nu \tag{53}$$

where $\alpha_{0\pm}$, α_ν are defined by (39) and (42) respectively. This function then corresponds to a situation in which some (abstract) operator is defined to have the spectral density

$$\begin{aligned}\frac{d\rho_0(\nu)}{\nu}&=\frac{d\nu}{N(\nu)\alpha_\nu}\qquad -1<\nu<1\\ &=\frac{\alpha_{0+}\delta(\nu-\nu_{0+})d\nu}{N_{0+}}-\frac{\alpha_{0-}\delta(\nu-\nu_{0-})d\nu}{N_{0+}}\qquad |\nu|>1.\end{aligned} \tag{54}$$

Thus, one may use $f_0(\mu\to\mu_0)$, in analogy with inverse scattering, as a sort of higher order comparison potential. This offers an alternate method of dealing with the direct problem when the Legendre expansion is not desirable in the main transport equation when $f(\mu\to\mu_0)$ is highly anisotropic.

For the purpose of the direct problem we now calculate the coefficients $A_{0\pm}(\mu)$ and $A_\nu(\mu)$. Thereby we obtain a relation between the basis used by Case[2] and the Legendre polynomials

$$A_\nu(\mu)=\int_{-1}^{1}d\mu'\mu'\,\phi_\nu(\mu')S_0(\mu',\mu). \tag{34b}$$

From Eq. (16) we have

$$S_0(\mu',\mu)=\sum_{n=0}^{\infty}\frac{2n+1}{2}\,P_n(\mu')q_n(\mu). \tag{55}$$

Solving Eq. (18) for $q_n(\mu)$ and substituting that in (55) yields

$$S_0(\mu',\mu)=\sum_{n=0}^{\infty}(2n+1)\frac{f_n}{1-cf_n}\,P_n(\mu')P_n(\mu). \tag{56}$$

From (34b) and (56), we get

$$A_\nu(\mu)=\sum_{n=0}^{\infty}(2n+1)\frac{f_n}{1-cf_n}\,P_n(\mu)\nu(W_n(\nu)-c\delta_{n0}), \tag{57}$$

where

$$W_n(\nu)=\int_{-1}^{1}d\mu\,\phi_\nu(\mu)P_n(\mu) \tag{58}$$

and used the fact that

$$\mu\phi_\nu(\mu)=\nu\phi_\nu(\mu)-\nu c/2. \tag{59}$$

Actually, $W_n(\nu)$ are a special case of the orthogonal polynomials obtained from the transport equation by direct expansion of $f(\mu'\to\mu)$ in terms of Legendre polynomials. In other words, when $f_n=\delta_{n0}$, we have from (58), (59) and the recurrence relation

$$(2n+1)\mu P_n(\mu)=(n+1)P_{n+1}(\mu)+nP_{n-1}(\mu) \tag{60}$$

the three-term pure recurrence relation for $W_n(\nu)$:

$$(n+1)W_{n+1}(\nu)+nW_{n-1}(\nu)=(2n+1)(1-c\delta_{n0})\nu W_n(\nu),\quad n>0 \tag{61}$$

with $W_0(\nu)=1$ and $W_{-1}(\nu)=0$. That Eq. (61) is a pure three-term recurrence relation implies that $W_n(\nu)$ are orthogonal and vice versa. But, more importantly, $W_n(\nu)$ can be easily

calculated either from Eq. (61) or from its explicit representation (58). All these considerations apply for $A_{0\pm}(\mu)$ by merely replacing ν by $\pm\nu_0$, where ν_0 is the zero of the dispersion function (28e). We shall defer any further discussion of the direct problem until Part 2, which will be a sequel to this paper.

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A variational principle for transport theory[a]

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A maximal variational principle is used to construct an infinite medium Green's function for treating the boundary value problems of the linear transport theory (neutron and radiative). For the neutrons we consider the one-speed case and correspondingly for the radiative transfer the monochromatic case. The scattering properties of the medium are presumed to be dependent on the relaxation length. Thus, for the neutrons the secondary production function depends on the neutron's relaxation length and for the radiative transfer the albedo for single scattering is dependent on the optical depth. These two functions are kept arbitrary so that a large class of problems can be covered. The basic principle involves a functional which is an absolute maximum when the trial function is an exact solution of an integral equation of the Fredholm type. The kernel of the integral equation is required to satisfy certain symmetry and boundedness properties. We also exhibit an interesting relation between the absolute maximal and Schwinger's stationary variational principles, which in general is neither a maximum nor a minimum.

1. INTRODUCTION

A variational principle for the solution of integral equations with probability kernels was used by Demarcus[1] to solve the problem of Knudsen flow. It involves a functional which is an absolute maximum when the trial function is an exact solution of the transport equation. This variational principle is a special case of one discussed, for example in Ref. 2, for which a functional is an absolute maximum when the trial function is an exact solution of an integral equation of the Fredholm type, such that the kernel is required to satisfy certain symmetry and boundedness properties. The more general variational principle has already been used for the Gel'fand—Levitan equation.[3,4] In the present paper we use the variational principle to provide insights into solutions of unsolved problems of transport theory such as radiative or neutron transfer in inhomogeneous media.

The abstract principle is discussed for example in Mikhlin's book[2] which we follow. It has been utilized by Demarcus[1] and much later by Stokes and Demarcus.[5] Our work differs from that of the others in the fundamental sense that we use the principle to construct the approximate Green's function and to estimate the spectral function of the transport operator for the trial function chosen. Greater flexibility is thus achieved with respect to boundary conditions.

2. THE VARIATIONAL PRINCIPLE

The basic principle underlying the variational approach (see Ref. 2) is, that given the Fredholm integral equation

$$f(x) = \phi(x) + \int_a^b dy\, K(x,y) f(y), \tag{1}$$

the functional

$$F[n] = \int_a^b dx\, n(x)[2\phi(x) + \int_a^b dy\, K(x,y) n(y) - n(x)] \tag{2}$$

is an absolute maximum when $n(x) = f(x)$, for the kernel $K(x,y)$ which satisfies the following conditions of symmetry and boundedness:

(a) the kernel $K(x,y)$ is symmetric in x, y,

(b) $K(x,y) \geq 0$ for $a \leq x \leq b$ and $a \leq y \leq b$,

(c) $\int_a^b dx K(x,y) < 1$ for $a \leq x \leq b$.

For most transport problems it turns out that the integral equation that one needs to consider (for example for an infinite medium Green's function) is not Eq. (1) but

$$f(x,x_0) = \phi(x,x_0) + \int_a^b dx'\, K(x,x') f(x',x_0), \tag{3}$$

where x_0 is an additional parameter. A further, rather pleasant, property that is encountered is when the inhomogeneous term $\phi(x,x_0)$ is the same as $K(x,x_0)$. Then Eq. (3) is in a symmetric form. That is, when

$$\phi(x,x_0) = K(x,x_0),$$

Eq. (3) becomes

$$f(x,x_0) = K(x,x_0) + \int_a^b dx'\, K(x',x) f(x',x_0), \tag{4}$$

and the general functional to be considered is

$$F[n] = \int_a^b dx\, n(x,x_0)[2K(x,x_0) + \int_a^b dx' K(x,x') n(x',x_0) - n(x,x_0)], \tag{5}$$

which is an absolute maximum when $n(x,x_0) = f(x,x_0)$ (c.f. Refs. 2 and 3). The resulting value of the functional is then

$$F[f] = \int_a^b dx\, f(x,x_0) K(x,x_0). \tag{6}$$

Using Eq. (4) in Eq. (6) to eliminate the integral we get

$$F[f] = \lim_{x \to x_0} [f(x,x_0) - K(x,x_0)]. \tag{7}$$

Thus, for an exact solution of the integral equation (4), the absolute maximum value of the functional is the difference of boundary values of $f(x,x_0)$ and the kernel $K(x,x_0)$ as x approaches x_0 from either side of x_0. Two further points are worth noting. First, in most tran-

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sport problems of interest the density function $(f(x,x_0))$ and the scattering kernel $(K(x,x_0))$ are singular at $x=x_0$. However, the difference of the two functions at $x=x_0$ is bounded from above. Furthermore, as we shall see later, the value of that difference is related to the spectral density of the transport operator corresponding to the choice of the trial function [for example see Eq. (57) in Sec. 3]. It is precisely for such a relation and the maximal property of the functional that it is possible to obtain the best estimates of the complete spectrum of the transport operator and hence the Green's function.

Now we consider an application of this formulation to the problems of radiative transfer in a vertically inhomogeneous atmosphere where the albedo for single scattering is an arbitrary function of the optical depth. The same application is valid, aside from changing of names, for the neutron transport in inhomogeneous media if the corresponding albedo for single scattering (or secondary production) of neutrons does not exceed unity anywhere in the given medium, i.e., for the subcritical problem.

3. APPLICATION TO PROBLEMS OF RADIATIVE TRANSFER AND ONE-SPEED NEUTRON TRANSPORT

We shall consider the problems of radiative transfer. Hence we use the corresponding appropriate terminology. The equation of radiative transfer for a plane parallel vertically inhomogeneous isotropically scattering atmosphere is

$$\mu\frac{\partial I}{\partial x}(x,\mu)+I(x,\mu)=\frac{\omega(x)}{2}\int_{-1}^{1}d\mu'\,I(x,\mu')+Q(x,\mu), \tag{8}$$

where μ is the cosine of the angle between the vertical axis and the direction of propagation of radiation, x is the optical depth, $\omega(x)$ is the albedo for single scattering, and $I(x,\mu)$ is the specific intensity. We have added a source term $Q(x,\mu)$ in case one is dealing with neutron transport. For problems of radiative transfer Q will be zero in general. Let the atmosphere be finite and optically bounded between $x=x_a$ and $x=x_b$. In order to set up the boundary value problem, we follow the standard procedure discussed in Refs. 6 and 7. Thus, consider a time reversed adjoint equation

$$\begin{aligned}-\mu\frac{\partial}{\partial x}G(x,-\mu\to x_0,-\mu_0)+G(x,-\mu\to x_0,-\mu_0)\\=\frac{\omega(x)}{2}\int_{-1}^{1}d\mu'\,G(x,-\mu'\to x_0,-\mu_0)\\+\frac{\sqrt{\omega(x)}}{2}\delta(x-x_0)\delta(\mu-\mu_0).\end{aligned} \tag{9}$$

Note that, here, the delta functions are weighted by $\sqrt{\omega(x)}/2$. Multiplying Eq. (8) by G and Eq. (9) by I, subtracting and integrating the difference with respect to μ from -1 to $+1$, and x from x_a to x_b, we get the result

$$\begin{aligned}I(x,\mu)=\frac{2}{\sqrt{\omega(x)}}\int_{-1}^{1}d\mu'\,\mu'[I(x_a,\mu')G(x_a,-\mu'\to x,-\mu)\\-I(x_b,\mu')G(x_b,-\mu'\to x,-\mu)]\\+\frac{2}{\sqrt{\omega(x)}}\int_{x_a}^{x_b}dx'\int_{-1}^{1}d\mu'\,Q(x',\mu')\\\times G(x',-\mu'\to x,-\mu).\end{aligned} \tag{10}$$

In particular, if

$$Q(x,\mu)=\frac{\sqrt{\omega(x)}}{2}\delta(x-x_1)\delta(\mu-\mu_1) \tag{10'}$$

and the medium is infinite, then $I(x,\mu)=\tilde{G}(x,\mu\to x_1,\mu_1)$ is a Green's function for the plane source at $x=x_1$ and $\mu=\mu_1$ satisfying Eq. (8). With suitable boundary conditions that both G and $\tilde{G}$ vanish at $x=\pm\infty$, we get from Eq. (10) the following reciprocity relation,

$$\sqrt{\omega(x)}\,G(x,-\mu\to x_1,-\mu_1)=\sqrt{\omega(x_1)}\,\tilde{G}(x_1,\mu_1\to x,\mu). \tag{10''}$$

One may use this reciprocity relation to replace G by $\tilde{G}$. However, we prefer to use G instead. For the finite medium and any given incident conditions for $I(x,\mu)$ at the boundaries (i.e., at $x=x_a$, $\mu>0$, $x=x_b$, $\mu<0$), the reflected and transmitted radiations are determined from the limits of the integral identity (10). In other words, the unknown quantities $I(x_a,\mu)$ for $\mu<0$ (the reflected intensity) and $I(x_b,\mu)$ for $\mu>0$ (the transmitted intensity) may be determined from the integral equations

$$\begin{aligned}I(x_a,\mu)=\frac{2}{\sqrt{\omega(x_a)}}\int_{-1}^{1}d\mu'\,\mu'[I(x_a,\mu')G^{+}(x_a,-\mu'\to x_a,-\mu)\\-I(x_b,\mu')G(x_b,-\mu'\to x_a,-\mu)]\\+\frac{2}{\sqrt{\omega(x_a)}}\int_{x_a}^{x_b}dx'\int_{-1}^{1}d\mu'\,Q(x',\mu')G(x',-\mu'\to x_a,-\mu),\\\mu>0\end{aligned} \tag{11}$$

and

$$\begin{aligned}I(x_b,\mu)=\frac{2}{\sqrt{\omega(x_b)}}\int_{-1}^{1}d\mu'\,\mu'[I(x_a,\mu')G(x_a,-\mu'\to x_b,-\mu)\\-I(x_b,\mu')G^{-}(x_b,-\mu'\to x_b,-\mu)]\\+\frac{2}{\sqrt{\omega(x_b)}}\int_{x_a}^{x_b}dx'\int_{-1}^{1}d\mu'\,Q(x',\mu')\\\times G(x',-\mu'\to x_b,-\mu),\quad\mu<0,\end{aligned} \tag{12}$$

where $+(-)$ on G represents its boundary value as x approaches the boundary of the atmosphere from the right (left). Solution of the integral equations (11) and (12) is possible provided we can construct the Green's function from the adjoint Eq. (9). In fact the proof of the existence of solutions of integral equations of this type for the grey problem is guaranteed by construction of the unknown coefficients (see Ref. 6 for further discussion). We focus our attention on procedure for the construction of G for the simple (and obvious) reason that it is the central quantity required for any boundary value problems of this kind. We shall demonstrate here the use of the variational technique to obtain an approximate solution to that problem of de-

termining G. To do that, we first obtain an integral equations for G. A convenient way is to take the Fourier transform of Eq. (9) with respect to x. The result is

$$G(x,-\mu \to x_0,-\mu_0)$$
$$=\frac{1}{4\pi}\int_{-\infty}^{\infty} dk\,\frac{\exp(ikx)}{1-ik\mu}\int_{-\infty}^{\infty} dx'\,\exp(-ikx')\omega(x')\int_{-1}^{1} d\mu'$$
$$\times\, G(x',-\mu'\to x_0,-\mu_0)+\sqrt{\omega(x_0)}\,\delta(\mu-\mu_0)$$
$$\times\,\frac{1}{4\pi}\int_{-\infty}^{\infty} dk\,\times\frac{\exp[ik(x-x_0)]}{1-ik\mu}. \tag{13}$$

For convenience in writing, denote the angular integral of the Green's function on the right-hand side of Eq. (13) by

$$H(x;x_0,\mu_0)=\int_{-1}^{1} d\mu\, G(x,-\mu\to x_0,-\mu_0). \tag{14}$$

Then we have

$$G(x,-\mu\to x_0,-\mu_0)$$
$$=\frac{1}{4\pi}\int_{-\infty}^{\infty} dk\,\frac{\exp(ikx)}{1-ik\mu}\int_{-\infty}^{\infty} dx'\,\omega(x')$$
$$\times \exp(-ikx')H(x';x_0,\mu_0)$$
$$+\sqrt{\omega(x_0)}\,\delta(\mu-\mu_0)\,\frac{1}{4\pi}\int_{-\infty}^{\infty} dk\,\frac{\exp[ik(x-x_0)]}{1-ik\mu}. \tag{15}$$

Clearly, if we can determine $H(x;x_0,\mu_0)$, then for any given $\omega(x)$, G is completely determined.

A. Integral equations for $H(x;x_0,\mu_0)$ and $\rho(x,x_0)$

By integrating Eq. (15) with respect to μ from -1 to $+1$ we obtain

$$H(x;x_0,\mu_0)=\frac{1}{\sqrt{\omega(x)}}\,\Psi(x;x_0,\mu_0)$$
$$+\frac{1}{\sqrt{\omega(x)}}\int_{-\infty}^{\infty} dx'\,K(x',x)\sqrt{\omega(x')}\,H(x';x_0,\mu_0), \tag{16}$$

where

$$\Psi(x;x_0,\mu_0)=\frac{1}{4\pi}\sqrt{\omega(x)\omega(x_0)}\int_{-\infty}^{\infty} dk\,\frac{\exp[ik(x-x_0)]}{1-ik\mu_0} \tag{17}$$

and

$$K(x',x)=\frac{1}{4\pi}\sqrt{\omega(x')\omega(x)}\int_{-\infty}^{\infty} dk\,\exp[ik(x-x')]\int_{-1}^{1}\frac{d\mu}{1-ik\mu}$$
$$=\tfrac{1}{2}\sqrt{\omega(x')\omega(x)}\int_{0}^{1}\frac{d\mu}{\mu}\,\exp(-|x-x'|/\mu)$$
$$=\tfrac{1}{2}\sqrt{\omega(x')\omega(x)}\,E_1(|x-x'|), \tag{18}$$

where $E_1(|x-x'|)$ is the exponential integral. Note that Ψ and K are related by

$$K(x,x_0)=\int_{-1}^{1} d\mu\,\Psi(x;x_0,\mu). \tag{19}$$

To complete the set of integral equations that we need, we define the density $\rho(x,x_0)$ by

$$\rho(x,x_0)=\int_{-1}^{1} d\mu\, H(x;x_0,\mu) \tag{20}$$

and from (16) we find that $\rho(x,x_0)$ satisfies the following integral equation,

$$\sqrt{\omega(x)}\,\rho(x,x_0)=K(x,x_0)+\int_{-\infty}^{\infty} dx'\,K(x',x)$$
$$\times\sqrt{\omega(x')}\,\rho(x',x_0), \tag{21}$$

where we have used the integral relation (19) between Ψ and K. If we let

$$H_S(x;x_0,\mu_0)=\sqrt{\omega(x)}\;H(x;x_0,\mu_0) \tag{22}$$

and

$$\rho_S(x,x_0)=\sqrt{\omega(x)}\,\rho(x,x_0), \tag{23}$$

then the integral equations (16) and (21) assume symmetric looking forms:

$$H_S(x;x_0,\mu_0)=\Psi(x;x_0,\mu_0)$$
$$+\int_{-\infty}^{\infty} dx'\,K(x',x)H_S(x';x_0,\mu_0) \tag{24}$$

and

$$\rho_S(x,x_0)=K(x,x_0)+\int_{-\infty}^{\infty} dx'\,K(x',x)\rho_S(x',x_0). \tag{25}$$

B. Functionals for $H_S(x;x_0,\mu_0)$ and $\rho_S(x,x_0)$

Consider the functionals

$$F_1[N]=\int_{-\infty}^{\infty} dx\,N(x;x_0,\mu_0)[2\Psi(x;x_0,\mu_0)-N(x;x_0,\mu_0)$$
$$+\int_{-\infty}^{\infty} dx'\,K(x',x)N(x';x_0,\mu_0)] \tag{26}$$

and

$$F[n]=\int_{-\infty}^{\infty} dx\,n(x,x_0)[2K(x,x_0)-n(x,x_0)$$
$$+\int_{-\infty}^{\infty} dx'\,K(x',x)n(x',x_0)]. \tag{27}$$

Since the kernel $K(x',x)$ in both integral equations (24) and (25) satisfies the three properties listed below Eq. (2), we conclude that functionals $F_1[N]^{\flat}$ and $F[n]$ are absolute maxima for exact solutions of those integral equations (see Refs. 3 and 4 for the proof). Thus, for

$$N(x;x_0,\mu_0)=H_S(x;x_0,\mu_0)$$

and

$$n(x,x_0)=\rho_S(x,x_0),$$

the values of the corresponding functionals are

$$F_1[H_S]=\int_{-\infty}^{\infty} dx\,\Psi(x;x_0,\mu_0)H_S(x;x_0,\mu_0) \tag{28}$$

and

$$F[\rho_S]=\int_{-\infty}^{\infty} dx\,K(x,x_0)\rho_S(x,x_0). \tag{29}$$

Further use of Eq. (25) in Eq. (29) yields

$$F[\rho_S]=\lim_{x\to x_0}[\rho_S(x,x_0)-K(x,x_0)]. \tag{30}$$

Equation (30) is a rather surprising result for the reason that the value of the functional for the density bears such a simple relation to the density $\rho_S(x,x_0)$ and the scattering kernel $K(x,x_0)$. This is, in the strict sense, a relation between the direct and the inverse problem. For the Gel'fand—Levitan equation (c.f. Ref. 3) we discovered a relation identical to Eq. (30) where it turned out that the result can be translated into a theorem about the area under a curve to a given point x when considered as a functional of $n(x,x_0)$. This curve was given by the scattering potential to the given point when the functional took on its maximum value. In that case the functional may thus be considered as a method of obtaining the scattering potential from the spectral data through the variational technique. However, for the transport problems we are considering such a simple interpretation, is not possible. For one reason, we note that both $\rho_S(x,x_0)$ and $K(x,x_0)$ are singular at $x=x_0$. However, in the limit when $x\to x_0$ the difference of the boundary values of $\rho_S(x,x_0)$ and $K(x,x_0)$ is bounded. It would be of value to determine that limit.

C. Value of the functional $F[\rho_S]$

Clearly, it is of crucial importance to know the value of the functional $F[\rho_S]$ for the exact solution of the integral equation (27). The reason is simply that for any sequence of trial functions, the values of the corresponding functionals must approach the absolute maximum value obtained for the exact solution. One can then estimate the absolute error introduced for any particular choice of a trial function. Of course, in order to find $F[\rho_S]$, as given by Eq. (30), we must know the exact form of the density $\rho_S(x,x_0)$ (which in fact is the unknown). However, since the functional is the boundary value of the difference between $\rho_S(x,x_0)$ and $K(x,x_0)$ when x approaches x_0, we may take advantage of that to construct a pseudo Green's function, such that if $\rho_g(x,x_0)$ is the corresponding pseudo density, the difference $\rho_S(x,x_0)-\rho_g(x,x_0)$ in the limit $x\to x_0$ is small. In other words we require that $\rho_g(x,x_0)$ be singular in the same manner as $\rho_S(x,x_0)$ at $x=x_0$ plus a (possible) small correction term which is nonsingular at $x=x_0$ for all values of x_0. We proceed as follows: Consider Eq. (9) for the Green's function

$$\mu\frac{\partial}{\partial x}G(x,\mu\to x_0,\mu_0)+G(x,\mu\to x_0,\mu_0)$$
$$=\frac{\omega(x)}{2}\int_{-1}^{1}d\mu'\,G(x',\mu'\to x_0,\mu_0) \tag{31}$$
$$+\frac{\sqrt{\omega(x)}}{2}\,\delta(x-x_0)\delta(\mu-\mu_0),$$

where for convenience we have changed the signs of μ and μ_0. Now consider a pseudo Green's function which satisfies

$$\mu\frac{\partial}{\partial x}G_g(x,\mu\to x_0,\mu_0)+G_g(x,\mu\to x_0,\mu_0)$$
$$=\frac{\omega(x_0)}{2}\int_{-1}^{1}d\mu'\,G_g(x,\mu'\to x_0,\mu_0) \tag{32}$$
$$+\frac{\sqrt{\omega(x)}}{2}\,\delta(x-x_0)\delta(\mu-\mu_0).$$

Note that ω associated with the first term on the right-hand side in Eq. (32) is a function of x_0 (the source point). If we let

$$L(x,\mu\to x_0,\mu_0)=G(x,\mu\to x_0,\mu_0)-G_g(x,\mu\to x_0,\mu_0), \tag{33}$$

then subtracting Eq. (32) from Eq. (31) we find that the difference $L(x,\mu\to x_0,\mu_0)$ satisfies the following equation,

$$\mu\frac{\partial}{\partial x}L(x,\mu\to x_0,\mu_0)+L(x,\mu\to x_0,\mu_0)$$
$$=\frac{\omega(x)}{2}\int_{-1}^{1}d\mu'\,L(x,\mu'\to x_0,\mu_0)$$
$$+\frac{\omega(x)-\omega(x_0)}{2}\int_{-1}^{1}d\mu'\,G_g(x,\mu'\to x_0,\mu_0). \tag{34}$$

In what follows, we assume that $\omega(x)$ is either a continuous function or can be approximated by a suitable continuous function. If ω were independent of x, then $L(x,\mu\to x_0,\mu_0)$ would be the infinite medium Green's function without sources. In consequence, with appropriate boundary conditions at $x=\pm\infty$, the only solution of Eq. (34) would be the trivial one, i.e., $L=0$, as one would expect. In that case G would coincide with G_g and the value of the functional [as given by Eq. (30)] may be readily shown to be proportional to the spectral density. For an arbitrary $\omega(x)$, the last term in Eq. (34) provides the "source" for $L(x,\mu\to x_0,\mu_0)$ and, therefore, that equation may have a nontrivial solution. However, we are only interested in the behavior of L near and at $x=x_0$. The first simple conclusion is that L is continuous everywhere on the real axis. This follows from the jump condition. Thus, if we integrate Eq. (34) from left to right of $x=x_0$, we get

$$\mu[L^+(x_0,\mu\to x_0,\mu_0)-L^-(x_0,\mu\to x_0,\mu_0)]=0, \tag{35}$$

where we have assumed that $\omega(x)$ is continuous and

$$L^\pm=\lim_{\substack{x\to x_0\ (\text{Right of } x_0)\\ (\text{Left of } x_0)}}L(x,\mu\to x_0,\mu_0).$$

Continuity of L also implies that its derivative is also continuous at $x=x_0$. This follows from simply taking the difference of boundary values of Eq. (34) as $x\to x_0$ from both sides of x_0. If we solve Eq. (35) for μ, then the most general solution is

$$L^+(x_0,\mu\to x_0,\mu_0)-L^-(x_0,\mu\to x_0,\mu_0)=\lambda_0\delta(\mu). \tag{36}$$

But, from Eq. (34) for $\mu=0$, we get

$$L(x,0\to x_0,\mu_0)=\frac{\omega(x)}{2}\int_{-1}^{1}d\mu'\,L(x,\mu'\to x_0,\mu_0)$$
$$+\frac{\omega(x)-\omega(x_0)}{2}\int_{-1}^{1}d\mu'\,G_g(x,\mu'\to x_0,\mu_0). \tag{37}$$

Thus,

$$L^+(x_0,0\to x_0,\mu_0)-L^-(x_0,0\to x_0,\mu_0)=0,$$

so that $\lambda_0\equiv 0$ in Eq. (36). We have, therefore, shown that $L(x,\mu\to x_0,\mu_0)$ is continuous everywhere and so is its derivative.

We now proceed as before as we did with Eq. (9) and obtain the integral equation for $\rho_L(x, x_0)$ defined by

$$\rho_L(x, x_0) = \int_{-1}^{1} d\mu \int_{-1}^{1} d\mu_0\, L(x, \mu \to x_0, \mu_0). \tag{38}$$

The integral equation satisfied by $\rho_L(x, x_0)$ is

$$\rho_L(x, x_0) = \tfrac{1}{2}\int_{-\infty}^{\infty} dx'\, E_1(|x' - x|)\omega(x')\rho_L(x', x_0) + \tfrac{1}{2}\int_{-1}^{\infty} dx'\, E_1(|x' - x|)[\omega(x') - \omega(x_0)] \times \rho_g(x', x_0), \tag{39}$$

where

$$\rho_g(x, x_0) = \int_{-1}^{1} d\mu \int_{-1}^{1} d\mu_0\, G_g(x, \mu \to x_0, \mu_0) \tag{40}$$

is the pseudo density and $E_1(|x' - x|)$ is the exponential integral,

$$E_1(|x' - x|) = \int_0^1 \frac{d\mu}{\mu} \exp(-|x' - x|/\mu). \tag{41}$$

Since $\rho_L(x, x_0)$ is continuous at $x = x_0$, we get

$$\rho_L(x_0, x_0) = \tfrac{1}{2}\int_{-\infty}^{\infty} dx'\, E_1(|x' - x_0|)\omega(x')\rho_L(x', x_0) = \tfrac{1}{2}\int_{-\infty}^{\infty} dx'\, E_1(|x' - x_0|)[\omega(x') - \omega(x_0)]\rho_g(x', x_0). \tag{42}$$

Now we make the approximation for the first integral on the right-hand side of Eq. (42). The largest contribution that the exponential integral $E_1(|x' - x_0|)$ makes is near $x' = x_0$. In fact E_1 is weakly singular (logarithmically) there and decays rather rapidly away from x_0. In analogy with the method of stationary phases, we assume that $\rho_L(x', x_0)$ varies slowly near $x' = x_0$ and, therefore, may be taken out of the integral. Equation (42) then gives us

$$\rho_L(x_0, x_0) = \tfrac{1}{2}\{\int_{-\infty}^{\infty} dx'\, E_1(|x' - x_0|)[\omega(x') - \omega(x_0)] \times \rho_g(x', x_0)\}/[1 - \tfrac{1}{2}\int_{-\infty}^{\infty} dx'\, E_1(|x' - x_0|)\omega(x')]. \tag{43}$$

But, by definition [see Eqs. (33), (38), and (40)]

$$\sqrt{\omega(x_0)}\, \rho_L(x_0, x_0) = \lim_{x \to x_0} [\sqrt{\omega(x)}\rho(x, x_0) - \sqrt{\omega(x_0)}\, \rho_g(x, x_0)]$$

therefore,

$$\sqrt{\omega(x_0)}\, \rho_L(x_0, x_0) = \lim_{x \to x_0} [\sqrt{\omega(x)}\, \rho(x, x_0) - K(x, x_0) - \sqrt{\omega(x_0)}\rho_g(x, x_0) + K(x, x_0)], \tag{44}$$

where $K(x, x_0)$ is, by definition (18), given by

$$K(x, x_0) = \tfrac{1}{2}\sqrt{\omega(x)\omega(x_0)}\, E_1(|x - x_0|). \tag{45}$$

Since, previously we defined $\rho_S(x, x_0) = \sqrt{\omega(x)}\rho(x, x_0)$ [see definition (23)] and now we define

$$\rho_{gS}(x, x_0) = \sqrt{\omega(x_0)}\, \rho_g(x, x_0), \tag{46}$$

we have from Eqs. (30) and (44) the value of the functional

$$F[\rho_S] = \lim_{x \to x_0} [\rho_S(x, x_0) - K(x, x_0)]$$

approximately equal to

$$F[\rho_S] \cong \lim_{x \to x_0} [\rho_{gS}(x, x_0) - K(x, x_0)] + \sqrt{\omega(x_0)}\, \rho_L(x_0, x_0), \tag{47}$$

where $\rho_L(x_0, x_0)$ is given by Eq. (43).

Here we have assumed that one can construct the Green's function $G_g(x, \mu \to x_0, \mu_0)$ by solving Eq. (32) and that the limit in Eq. (47) exists and is bounded. We answer that question by merely noting that the method of obtaining solution of Eq. (32) is in exact parallel to the case where the medium is homogeneous. In fact, following Case and Zweifel [see Eq. (11), p. 97, in Ref. 6], we find that

$$G_g(x, \mu \to x_0, \mu_0) = \pm \frac{\sqrt{\omega(x_0)}}{2}\left(\frac{\phi_{0\pm}(\mu)\phi_{0\pm}(\mu_0)\exp(-|x - x_0|/\nu_0)}{N_{0\pm}} + \int_0^1 \frac{d\nu}{N(\pm\nu)}\, \phi_{\pm\nu}(\mu)\phi_{\pm\nu}(\mu_0)\exp(-|x - x_0|/\nu)\right), \tag{48}$$

where $+(-)$ is for $x > x_0$ $(x < x_0)$, and $\phi_{0\pm}(\mu)$, $\phi_\nu(\mu)$ are discrete and continuum Case eigenfunctions given by

$$\phi_{0\pm}(\mu) = \frac{\nu_0\omega(x_0)}{2(\pm\nu_0 - \mu)}, \tag{49}$$

$$\phi_\nu(\mu) = \frac{\nu\omega(x_0)}{2}\, P\frac{1}{\nu - \mu} + \lambda(\nu)\delta(\nu - \mu), \tag{50}$$

$$\lambda(\nu) = 1 - \nu\omega(x_0)\tanh^{-1}\nu, \tag{51}$$

$$N_{0\pm} = \frac{\omega(x_0)}{2}\, \nu_0^3\left(\frac{\omega(x_0)}{\nu_0^2 - 1} - \frac{1}{\nu_0^2}\right), \tag{52}$$

$$N(\nu) = \nu[(1 - \omega(x_0)\nu\tanh^{-1}\nu)^2 + \omega^2(x_0)\pi^2\nu^2/4], \tag{53}$$

and ν_0 is the zero of the dispersion function, i.e.,

$$1 - \frac{\omega(x_0)\nu_0}{2}\ln\left(\frac{\nu_0 + 1}{\nu_0 - 1}\right) = 0. \tag{54}$$

We remark here that Cases's discrete spectrum obtained from the solution of Eq. (54) is in fact now a "Regge" type trajectory determined by the functional form of $\omega(x_0)$. Further, the functional $F[\rho_S]$ will also follow a trajectory, which we will call "maximal trajectory." It is now a simple matter to find that maximal trajectory. This we do by noting that $\phi_{0\pm}(\mu)$ and $\phi_\nu(\mu)$ are normalized to unity, i.e.,

$$\int_{-1}^{1} d\mu\, \phi_{0\pm}(\mu) = \int_{-1}^{1} d\mu\, \phi_\nu(\mu) = 1 \tag{55}$$

so that

$$\rho_{gS}(x, x_0) = \frac{\omega(x_0)}{2}\left(\frac{\exp(-|x - x_0|/\nu_0)}{N_{0\pm}} + \int_0^1 \frac{d\nu}{N(\nu)}\exp(-|x - x_0|/\nu)\right). \tag{56}$$

Hence

$$F[\rho_S] \cong \frac{\omega(x_0)}{2}\left[\frac{1}{N_{0+}} + \lim_{x\to 0}\int_0^1 d\nu \exp(-|x|/\nu)\left(\frac{1}{N(\nu)} - \frac{1}{\nu}\right)\right]$$

$$+ \sqrt{\omega(x_0)}\,\rho_L(x_0, x_0), \qquad (57)$$

where, as a reminder, $\rho_L(x_0, x_0)$ is given by Eq. (43). From Eq. (57) we see that the functional is bounded. In the lowest approximation the continuum part, involving the integral, may be neglected; because, for small x the largest contribution from the integral involving $N(\nu)$ comes from near $\nu = 0$, where $N(\nu) \sim \nu$. Hence

$$F_0[\rho_S] = \frac{\omega(x_0)}{2N_{0+}} + \sqrt{\omega(x_0)}\,\rho_L(x_0, x_0), \qquad (58)$$

where we have subscripted the functional with zero to indicate the zeroth approximation. We close this section with one more remark. Here we have demonstrated the use of a maximal variational principle to construct an appropriate Green's function for inhomogeneous media in dealing with problems of radiative and neutron transport. We have made no attempt to solve any actual problem for a real medium. We hope to make that a topic for the next paper. However, we present a simple example in Sec. 14, where we consider a trial function involving G_g.

We also wish to point out that the same variational principle seems to be applicable to wider variety of problems in theoretical physics, viz., kinetic theory of gases, plasma physics (such as the Vlasov equation), and electron transport in the upper atmosphere.

4. EXAMPLE

For the trial Green's function we take the simplest form which corresponds to the *scaling* of G_g, i.e., we let

$$G_T(x, \mu \to x_0, \mu_0) = \alpha\,\frac{\omega(x_0)}{\omega(x)}\,G_g(x, \mu \to x_0, \mu_0), \qquad (59)$$

where α is a parameter to be determined from maximization of the functional (27). Now it follows that the trial density function is

$$\rho_T(x, x_0) = \alpha\,\frac{\omega(x_0)}{\omega(x)}\,\rho_g(x, x_0). \qquad (60)$$

It is seen, by replacing $\omega(x)$ by $\omega(x_0)$ everywhere in Eq. (25), that $\rho_{gS} = \sqrt{\omega(x_0)}\,\rho_g$ satisfies the following integral equation

$$\rho_{gS}(x, x_0) = K_g(x, x_0) + \int_{-\infty}^{\infty} dx'\,K_g(x', x)\rho_{gS}(x', x_0), \qquad (61)$$

where the kernel K_g is

$$K_g(x, x_0) = \frac{\omega(x_0)}{2}\,E_1(|x - x_0|). \qquad (62)$$

Now in the expression (27) for the density functional we set $n = \rho_{TS} = \sqrt{\omega(x)}\,\rho_T(x, x_0)$ and compute the value of the functional. One may readily show via use of integral equation (61) that the value of the functional is

$$F[\rho_{TS}] = \alpha(2 - \alpha)\tilde{F}[\rho_{gS}] - \alpha^2 W(x_0), \qquad (63)$$

where by definition

$$\tilde{F}[\rho_{gS}] = \int_{-\infty}^{\infty} dx\,K_g(x, x_0)\rho_{gS}(x, x_0) \qquad (64)$$

and

$$W(x_0) = \int_{-\infty}^{\infty} dx\left(\frac{\omega(x_0)}{\omega(x)} - 1\right)\rho^2_{gS}(x, x_0). \qquad (65)$$

Setting

$$\frac{\partial F}{\partial \alpha}[\rho_{TS}] = 0$$

we get

$$\alpha(x_0) = \frac{1}{1 + W(x_0)/\tilde{F}[\rho_{gS}]}. \qquad (66)$$

Note that $\tilde{F}[\rho_{gS}]$ is the exact maximum value of the functional for the pseudo grey medium. The value of the functional $F[\rho_{TS}]$ is given by

$$F[\rho_{TS}] = \frac{\tilde{F}^2[\rho_{gS}]}{\tilde{F}[\rho_{gS}] + W(x_0)}. \qquad (67)$$

To test the accuracy of the trial function chosen, we compute

$$\epsilon(x_0) = \frac{F[\rho_{TS}] - \tilde{F}[\rho_{gS}]}{F[\rho_{TS}]} \qquad (68)$$

and see how far $\epsilon(x_0)$ is away from zero. Thus, for the present example considered $\epsilon(x_0)$ represents the local deviation from the grey approximation.

5. A RELATION BETWEEN THE MAXIMAL AND SCHWINGER'S VARIATIONAL PRINCIPLES

In the previous sections we have illustrated the basic ideas underlying the maximal variational principle as applied to transport theory. Because the functional used is an absolute maximum for the exact solution of the appropriate transport equation, it has the advantage that, in principle at least, one can obtain the estimate of absolute errors for any set of trial functions used. We have already provided an approximate uniform bound to the value of the functional which for most situations should be useful. By contrast other variational principles, such as Schwinger's or Rayleigh Ritz principles, which require stationarity only, the estimate of errors for a set of trial functions for the entire medium (i.e., the uniform error) is usually not feasible. In this sense those variational principles are rather limited and often very cumbersome when applied to real problems. However, it would be of pedagogical value to show a relation between the maximal principle which we have used and Schwinger's stationary principle.

Schwinger's variational principle[8] states that, given the Fredholm integral equation

$$y(r, r_0) = K(r, r_0) + \int_0^{\infty} dr'\,K(r', r)y(r'), \qquad (69)$$

where the kernel $K(r', r)$ is symmetric, then the functional

$$S[n] = \left\{\int_0^{\infty} dr\,n(r, r_0)[n(r, r_0) - \int_0^{\infty} dr'\,K(r', r)n(r', r_0)]\right\} /$$

$$\left[\int_0^{\infty} dr\,n(r, r_0)K(r, r_0)\right]^2 \qquad (70)$$

is *stationary* when $n(r,r_0)=y(r,r_0)$. The value of the functional is then

$$S[y]=\frac{1}{\int_0^\infty dr\,K(r,r_0)y(r,r_0)}. \tag{71}$$

Let

$$N[r_0]=\int_0^\infty dr\,K(r,r_0)n(r,r_0). \tag{72}$$

By comparing Schwinger's functional (70) and the maximal functional $F[n]$ (5), we conclude that the two functionals bear the following relationship with each other,

$$F[n]=2N(r_0)-S[n]N^2(r_0). \tag{73}$$

For the exact solution of Eq. (69), we have already from Eq. (70) that $S[y]=1/N(r_0)$. Therefore,

$$F[y]S[y]=1. \tag{74}$$

In other words, for the exact solution, the product of maximal functional and Schwinger's stationary functional is unity.

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DIRECT-INVERSE PROBLEMS IN TRANSPORT THEORY, THE INVERSE ALBEDO PROBLEM FOR A FINITE MEDIUM

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ABSTRACT

In this paper we deal with the inverse problem in radiative transfer, which is equally applicable to the neutron transport, for a finite homogeneous medium. We give a method for computing the scattering function and the albedo for single scattering. The solution is given in terms of Legendre expansion of the scattering function. A decomposition of the equation of transfer is also given in which the relation between the direct and the inverse problem is exhibited via the principles of invariance. A relation of this work with Case's method of approach is outlined. This work should be of practical value to the problems associated with remote sensing of the terrestrial atmosphere and the neutron piles.

1.0 INTRODUCTION

In an earlier paper by Kanal and Moses [1], an inverse problem in neutron (or radiative) transport theory was solved for an infinite medium containing a plane source emitting neutrons (or photons) in a direction whose cosine of the angle was fixed arbitrarily at $\mu = \mu_0$. The objective was to reconstruct the scattering kernel $f(\mu' \to \mu)$ (or the phase function) from the information obtained from the experiment which measured the neutron (or photon) column density for all angles; which is the zeroth moment of the angular density (or the specific intensity of the radiation field). In contrast the object of the usual direct problem is to find the distribution function (angular density of neutrons or the specific intensity of radiation) for a given scattering function. In this paper we wish to treat the inverse-albedo problem of radiative transfer in an atmosphere which is finite in the total optical depth scale. The direct albedo problem for a finite atmosphere is defined as follows: we are given an incident radiation field at $x = 0$, where x is the optical depth. Also given are the phase function, the albedo for single scattering and the reflection boundary condition at $x = x_b$, where x_b is the total optical thickness of the atmosphere. The problem is to determine the radiation field everywhere and, in consequence, the reflected component at $x = 0$ and the transmitted component at $x = x_b$. In contrast the inverse-albedo problem for the finite

atmosphere involves the construction of the phase function and the albedo for single scattering from the results of an experiment which measures a certain minimum set of suitable quantities. We discuss the nature of measurements that are required for the present treatment below.

In an earlier paper [2], K. M. Case gives an elegant solution of an inverse problem for an infinite medium containing an isotropic plane neutron source and constructs the scattering function from the spectral data of the transport operator. He requires measurements of the neutron density for all relaxation lengths, and relates the density to the spectral function. He, then proceeds to show by different routes (including the discretized version of the Gelfand-Levitan equation) how one may construct the coefficients in the Legendre polynomial expansion of the scattering function. In our treatment of the albedo-inverse problem, which physically differs from the infinite medium problem, the measurements that we require are different from the ones required by Case. In parallel with the infinite medium problem [1], we require the measurements to give information on the finite integral of the specific intensity, i.e., we require to know

$$\int_{x_1}^{x_2} dx\ I(x,\mu)$$

and $I(x_1,\mu)$, $I(x_2,\mu)$, where x_1 and x_2 are two arbitrary points in the optical depth scale. We shall then see how that quantity is related to the infinite medium problem via the princi-

ples of invariance [3]. We shall also demonstrate how one may actually construct the spectral data for the Case-model from our experiment and in consequence generate Case-eigenfunctions from that experiment.

A final remark is due here. In our previous problem [1], we used two different bases for the expansion of the phase function. One was the Legendre polynomia and the second was the complete set of Case's eigenfunctions [4] of the transport operator for the <u>isotropic</u> scattering medium. Both expansions have their inherent advantages. In the Legendre polynomial expansion, the inverse problem is exactly soluble and is particularly useful when the medium is not too anisotropically scattering. On the other hand, if the medium is highly anisotropically scattering, then the eigenfunction expansion is particularly suitable. For in that case we have a maximum variational principle which can be used (and in fact we have used [1]) to obtain a bounded estimate of the expansion coefficients. However, the primary aim of this paper is to relate the infinite medium solution to the inverse-albedo problem for a finite atmosphere by an alternate formulation of the principles of invariance. We shall address the albedo problem in radiative transfer.

2.0 INVERSE-ALBEDO PROBLEM FOR A FINITE ATMOSPHERE

Consider a finite medium optically bounded between $x = 0$ and $x = x_b$ as shown in Fig. 1. For the one-dimensional, rotationally invariant case, the standard equation of radiative transfer is [3],

$$\mu \frac{\partial I(x,\mu)}{\partial x} + I(x,\mu) = \frac{c}{2} \int_{-1}^{1} d\mu' \, f(\mu' \to \mu) \, I(x,\mu') \, . \quad (2.1)$$

Where μ is the cosine of the angle corresponding to the unit vector pointing in the direction of propagation of radiation, x is the optical depth, $I(x,\mu)$ is the specific intensity of the radiation field, c is the albedo for single scattering and $f(\mu' \to \mu)$ is the phase function. The phase function is normalized to unity so that

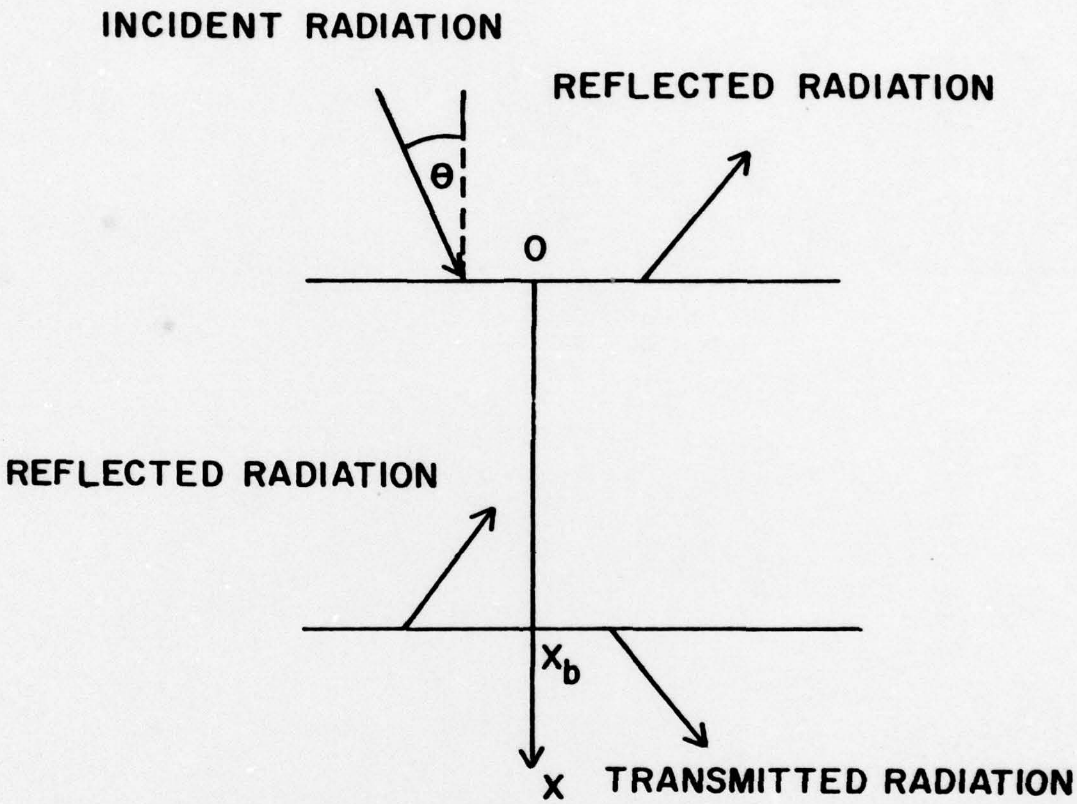
$$\frac{1}{2} \int_{-1}^{1} d\mu' \, f(\mu' \to \mu) = 1 \, . \quad (2.2)$$

Now imagine the finite medium, under consideration, imbedded in an infinite medium with the same scattering properties. Further, imagine a plane source of radiation at $x = x_0$ in the infinite medium emitting photons in a direction $\mu = \mu_0$. The equation of transfer for that case is

$$\mu \frac{\partial \Psi(x,\mu \to x_0,\mu_0)}{\partial x} + \Psi = \frac{c}{2} \int_{-1}^{1} d\mu' \, f(\mu' \to \mu) \, \Psi(x,\mu' \to x_0,\mu_0)$$

$$+ \delta(x-x_0) \, \delta(\mu-\mu_0) \, . \quad (2.3)$$

Let us consider the column density of the radiation field for the infinite medium so that,



CONFIGURATION FOR THE INVERSE-ALBEDO PROBLEM

FIGURE 1

$$M_0(\mu,\mu_0) = \int_{-\infty}^{\infty} dx\ \Psi(x,\mu\to x_0,\mu_0)\ . \tag{2.4}$$

Also, let us define the total emission term by

$$S_0(\mu,\mu_0) = \int_{-1}^{1} d\mu'\ f(\mu'\to\mu)\ M_0(\mu',\mu_0)\ . \tag{2.5}$$

With the appropriate boundary conditions at $x = \pm\infty$,

$$\Psi(\pm\infty,\mu\to x_0,\mu_0) = 0\ , \tag{2.6}$$

we find from Eq. (2.3) that

$$M_0(\mu,\mu_0) = \delta(\mu-\mu_0) + \frac{c}{2}\ S_0(\mu,\mu_0)\ . \tag{2.7}$$

By multiplying Eq. (2.7) on both sides with $f(\mu\to\mu_0)$ and integrating with respect to μ, we find that the total emission term $S_0(\mu,\mu_0)$ satisfies the integral equation of the Fredholm type:

$$S_0(\mu,\mu_0) = f(\mu\to\mu_0) + \frac{c}{2}\int_{-1}^{1} d\mu'\ f(\mu'\to\mu)\ S_0(\mu',\mu_0)\ . \tag{2.8}$$

This is an integral equation for $S_0(\mu,\mu_0)$ for the infinite medium whose solutions are discussed in ref. [1]. In the same manner as above, consider the measurement of the column density for the finite medium between any two points x_1, x_2:

$$M_f(\mu,\mu_1|x_1,x_2) = \int_{x_1}^{x_2} dx\ I(x,\mu)\ , \tag{2.9}$$

where μ_1 refers to the cosine direction of the incidence beam at $x = 0$. Integrating Eq. (2.1) from x_1 to x_2, we obtain

$$M_f(\mu,\mu_1|x_1,x_2) = \mu I(x_1,\mu) - \mu I(x_2,\mu) + \frac{c}{2}\ S_f(\mu,\mu_1|x_1,x_2) \tag{2.10}$$

where

$$S_f(\mu,\mu_1|x_1,x_2) = \int_{-1}^{1} d\mu' \; f(\mu'\to\mu) \; M_f(\mu',\mu_1|x_1,x_2) \; , \quad (2.11)$$

is the emission term for a slab of the atmosphere between $x = x_1$ and $x = x_2$. The question we ask is, what is the relation between the total emission term $S_0(\mu,\mu_0)$ and the emission term $S_f(\mu,\mu_1|x_1,x_2)$ for the slab, if we know the specific intensity $I(x,\mu)$ of radiation at the two points $x = x_1$ and $x = x_2$? From that knowledge, how do we construe the scattering properties of the finite medium? Answers to these questions are readily provided by the principles of invariance [3], stated as follows: "The law of scattering by a finite homogeneous atmosphere must be invariant to the addition (or subtraction) of layers of arbitrary thickness to (or from) the homogeneous atmosphere."

Let us see how we may apply this principle to solve the inverse albedo problem. Consider Eq. (2.10) and multiply it by $f(\mu\to\mu_2)$ on both sides. An integration with respect to μ, then yields

$$S_f(\mu,\mu_1|x_1,x_2) = \int_{-1}^{1} d\mu'\mu' \; f(\mu'\to\mu)[I(x_1,\mu') - I(x_2,\mu')]$$

$$+ \frac{c}{2}\int_{-1}^{1} d\mu' \; f(\mu'\to\mu) \; S_f(\mu',\mu_1|x_1,x_2) \quad (2.12)$$

where we have renamed the variables. As an intermediate step, expand $f(\mu'\to\mu)$ in terms of Legendre polynomia,

$$f(\mu'\to\mu) = \sum_{n=0}^{\infty} (2n+1) \; f_n \; P_n(\mu') \; P_n(\mu) \; . \quad (2.13)$$

Insert (2.13) in (2.12) and use the orthogonality property of $P_n(\mu)$ to obtain,

$$f_n = \frac{\int_{-1}^{1} d\mu\ P_n(\mu)\ S_f(\mu,\mu_1|x_1,x_2)}{\int_{-1}^{1} d\mu\ P_n(\mu)[2\mu(I(x_1,\mu)-I(x_2,\mu))+c\ S_f(\mu,\mu_1|x_1,x_2)]} . \tag{2.14}$$

Now in ref. [1] (see Eq. (18) in that reference), we obtained the following relation for an infinite atmosphere:

$$f_n = \frac{q_n(\mu_0)}{2P_n(\mu_0)+c\ q_n(\mu_0)} , \tag{2.15}$$

where

$$q_n(\mu_0) = \int_{-1}^{1} d\mu'\ P_n(\mu)\ S_0(\mu,\mu_0) , \tag{2.16}$$

so that

$$S_0(\mu,\mu_0) = \sum_{n=0}^{\infty} \frac{2n+1}{2}\ q_n(\mu_0)\ P_n(\mu) . \tag{2.17}$$

Now we apply the principles of invariance by equating the right-hand-sides of Eqs. (2.14) and (2.15). Transposing the denominator terms to the corresponding sides we obtain

$$\int_{-1}^{1} d\mu\ P_n(\mu)\ S_f(\mu,\mu_1|x_1,x_2)\ [2P_n(\mu_0) + c\ q_n(\mu_0)]$$

$$= \int_{-1}^{1} d\mu\ q_n(\mu_0)\ P_n(\mu)[2\mu(I(x_1,\mu)-I(x_2,\mu)) + c\ S_f(\mu,\mu_1|x_1,x_2)] . \tag{2.18}$$

Note that the second term in the bracket on this left-hand-side cancels the last term in the bracket on the right-hand-side. Now multiply Eq. (2.18) by (2n+1)/2 and sum over n.

Noting the completeness relation for the Legendre polynomia

$$\sum_{n=0}^{\infty} \frac{2n+1}{2} P_n(\mu)\, P_n(\mu_0) = \delta(\mu-\mu_0) \;, \tag{2.19}$$

and the definition (2.17) of $S_0(\mu,\mu_0)$, we get

$$S_f(\mu_0,\mu_1|x_1,x_2) = \int_{-1}^{1} d\mu\mu\; S_0(\mu,\mu_0)\,[I(x_1,\mu) - I(x_2,\mu)] \;. \tag{2.20}$$

Equation (2.20) represents a mathematical form of the principles of invariance relating the emission term for a finite slab embedded in an infinite atmosphere to its total emission term $S_0(\mu,\mu_0)$. Thus S_0 plays the role of a kind of Green's function. The symbol μ_1 is reserved for the direction of the incident beam at $x = 0$ for the finite atmosphere.

From Eqs. (2.14) and (2.10) we readily see that the expansion coefficients f_n can be obtained from the measurement $M_f(\mu,\mu_1|x_1,x_2)$ of the column density between x_1,x_2. Thus, we have

$$f_n = \frac{1}{c}\left[1 - \frac{\int_{-1}^{1} d\mu\; P_n(\mu)\; \mu(I(x_1,\mu) - I(x_2,\mu))}{\int_{-1}^{1} d\mu\; P_n(\mu)\; M_f(\mu,\mu_1|x_1,x_2)}\right] . \tag{2.21}$$

For later pruposes we introduce the quantity

$$g_n = 1 - cf_n \tag{2.22}$$

where,

$$g_n = \frac{\int_{-1}^{1} d\mu\; P_n(\mu)\; \mu(I(x_1,\mu) - I(x_2,\mu))}{\int_{-1}^{1} d\mu\; P_n(\mu)\; M_f(\mu,\mu_1|x_1,x_2)} \;. \tag{2.23}$$

For $n = 0$, $f_0 = 1$, we obtain the value of the albedo for single scattering c,

$$c = 1 - \frac{\int_{-1}^{1} d\mu\ \mu(I(x_1,\mu) - I(x_2,\mu))}{\int_{-1}^{1} d\mu\ M_f(\mu,\mu_1|x_1,x_2)} . \tag{2.24}$$

Principles of invariance, represented by Eq. (2.20) gives rise to an interesting decomposition of the equation of transfer (2.1). For in the limit $x_1 \to x_2$ ($x_2 = x$), Eq. (2.20) becomes

$$\lim_{x_1 \to x} \frac{S_f(\mu_0,\mu_1|x_1,x)}{x - x_1} = - \int_{-1}^{1} d\mu\mu\ S_0(\mu,\mu_0) \frac{\partial I(x,\mu)}{\partial x} \tag{2.25}$$

while from Eqs. (2.10) and (2.9) we obtain,

$$\frac{2}{c} [I(x,\mu_0) + \mu_0 \frac{\partial I(x,\mu_0)}{\partial x}] = \lim_{x_1 \to x} \frac{S_f(\mu_0,\mu_1|x_1,x)}{x - x_1} . \tag{2.26}$$

Hence from (2.25) and (2.26) we get

$$\mu \frac{\partial I(x,\mu)}{\partial x} + I(x,\mu) = - \frac{c}{2} \int_{-1}^{1} d\mu'\ S_0(\mu',\mu')\ \mu' \frac{\partial I(x,\mu')}{\partial x} . \tag{2.27}$$

Equating the right-hand-side of this equation to the right-hand-side of Eq. (2.1) we also obtain the relation

$$\int_{-1}^{1} d\mu'\ S_0(\mu',\mu)\ \mu' \frac{\partial I(x,\mu')}{\partial x} + \int_{-1}^{1} d\mu'\ f(\mu' \to \mu)\ I(x,\mu') = 0 . \tag{2.28}$$

It is clear that Eqs. (2.27) and (2.28) are completely equivalent to the original equation (2.1) of radiative transfer. However, we remark that Eq. (2.28) is strictly a relation between the direct and the inverse problems. Of course, one

would desire a converse relation to Eq. (2.20) in which the infinite medium measurement is related to the finite medium. Such a relation is readily obtained from Eq. (2.18), which gives the expansion coefficients $q_n(\mu_0)$ of $S_0(\mu,\mu_0)$ (see Eq. (2.17)). Thus,

$$q_n(\mu_0) = \frac{P_n(\mu_0) \int_{-1}^{1} d\mu\, P_n(\mu)\, S_f(\mu,\mu_1|x_1,x_2)}{\int_{-1}^{1} d\mu\, P_n(\mu)\, \mu(I(x_1,\mu) - I(x_2,\mu))} \tag{2.29}$$

or in terms of M_f,

$$q_n(\mu_0) = \frac{2}{c} P_n(\mu_0) \left[\frac{\int_{-1}^{1} d\mu\, P_n(\mu)\, M_f(\mu,\mu_1|x_1,x_2)}{\int_{-1}^{1} d\mu\, P_n(\mu)\, \mu(I(x_1,\mu) - I(x_2,\mu))} - 1 \right]. \tag{2.30}$$

Comparing (2.30) with (2.21) we find that

$$q_n(\mu_0) = P_n(\mu_0) \frac{2f_n}{1-cf_n} . \tag{2.31}$$

From this and Eq. (2.17) it clearly follows that

$$S_0(\mu,\mu_0) = \sum_{n=0}^{\infty} (2n+1) \frac{f_n}{1-cf_n} P_n(\mu)\, P_n(\mu_0) \tag{2.32}$$

where f_ns are obtained from the measurements as per Eq. (2.21). Thus, relations (2.20) and (2.32) represent a reciprocity between the measurements of a finite medium relating to the measurements of the infinite medium.

Next we shall consider the relationship between our process of measurement to the one considered by Case [2] and actually see how one may generate Case-eigenfunctions from the

measurements and also see how that leads to the construction of the spectral data.

3.0 RELATION OF INVERSE-ALBEDO PROBLEM TO CASE'S MODEL

For the sake of clarification some repetition of Case's work is essential here. In his model one has an infinite medium with a plane neutron source emitting neutrons isotropically at $x = 0$. The homogeneous equation of transfer is then,

$$\mu \frac{\partial \psi}{\partial x} + \psi = c \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} f_\ell P_\ell(\mu) \int_{-1}^{1} d\mu' P_\ell(\mu') \psi(\mu') \qquad (3.1C)$$

where the scattering kernel has been expanded in terms of Legendre polynomia and we have suffixed the equations with C to acknowledge Case's work. Looking for infinite medium solutions of this equation in the form $\phi_\nu(\mu) \exp(-x/\nu)$ one finds

$$(\nu-\mu) \phi_\nu(\mu) = \frac{c\nu}{2} M(\mu,\nu) \qquad (3.2C)$$

where

$$M(\mu,\nu) = \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell P_\ell(\mu) h_\ell(\nu) \qquad (3.3C)$$

with

$$h_\ell(\nu) = \int_{-1}^{1} d\mu \phi_\nu(\mu) P_\ell(\mu) \qquad (3.4C)$$

and

$$h_0 = 1 .$$

It has been shown that a complete set of eigenfunctions are

$$\phi_\nu(\mu) = \frac{1}{2} c\nu P[M(\mu,\nu)/(\nu-\mu)] + \lambda\delta(\nu-\mu) \qquad (3.5C)$$

and

$$\phi_{\nu i}(\mu) = \frac{1}{2} c\nu_i \, [M(\mu,\nu_i)/(\nu_i-\mu)] \tag{3.6C}$$

where

$$\lambda(\nu) = \frac{1}{2} [\Lambda^+(\nu) + \Lambda^-(\nu)] \tag{3.7C}$$

$$\Lambda(\nu) = 1 - \frac{1}{2} c\nu \int_{-1}^{1} d\mu \, \frac{M(\mu,\nu)}{\nu - \mu} . \tag{3.8C}$$

$\Lambda(\nu_i) = 0$, and +(-) represents the boundary value of Λ as the branch cut is approached from the top (bottom). Under the given assumptions, there are only a finite number of real, simple ν_i with $|\nu_i| > 1$. They occur in equal but opposite pairs.

These functions are orthogonal in the sense that

$$\int_{-1}^{1} d\mu\mu \, \phi_\nu(\mu) \, \phi_{\nu'}(\mu) = N(\nu) \, \delta(\nu-\nu') \tag{3.9C}$$

$$\int_{-1}^{1} d\mu\mu \, \phi_i(\mu) \, \phi_j(\mu) = N_i \, \delta(i,j) \tag{3.10C}$$

As an application of the orthogonality and completeness properties one can calculate total density corresponding to a unit plane source at $x = 0$. This is the solution of Eq. (3.1C) when the inhomogeneous term $\delta(x)/2\pi$ is added on the right. The result is

$$\Phi(x) = 2\pi \int_{-\infty}^{\infty} \frac{d\rho(\nu)}{\nu} \exp\left(-|x/\nu|\right) \tag{3.11C}$$

where,

$$\frac{d\rho(\nu)}{\nu} = \frac{d\nu}{N(\nu)} , \; -1 \le \nu \le 1 \tag{3.12C}$$

$$= \sum_i \frac{\delta(\nu-\nu_i)d\nu}{N_i} , \; |\nu| > 1 . \tag{3.13C}$$

Then Case states that, given measurements of $\Phi(x)$, one knows quite a bit about the spectral function $\rho(\nu)$.

The functions $h_\ell(\nu)$ obey a three terms pure recurrence relation

$$(\ell+1)\ h_{\ell+1}(\nu) + \ell h_{\ell-1}(\nu) = (2\ell+1)(1-cf_\ell)\ \nu h_\ell(\nu) = 0\ . \qquad (3.14C)$$

Case's version of the inverse problem is thus reduced to the following: Given the information about $d\rho(\nu)$, which Eq. (3.11C) provides, what is $g(n) = 1 - cf_n$?

Now Case has shown [4] that the normalization coefficients $N(\nu)$ and N_i in Eqs. (3.9C) and (3.10C) are related to the dispersion function $\Lambda(\nu)$ by

$$N(\nu) = \nu\ \Lambda^+(\nu)\ \Lambda^-(\nu) \qquad (3.15C)$$

and

$$N_{i\pm} = \pm\ \frac{c}{2}\ \nu_i^2\ M(\nu_i,\nu_i)\ \frac{\partial\Lambda}{\partial\nu}\ \Big|_{\nu\ =\ \nu_i} \qquad (3.16C)$$

respectively. Clearly therefore, the knowledge of the dispersion function is all one needs to generate $N(\nu)$, N_i and in consequence the spectral function, given by Eqs. (3.12C) and (3.13C) and all eigenfunctions. From Eq. (3.11C) it is not obvious as to how one may in practice actually obtain the information about the spectral function from the knowledge of $\Phi(x)$. However, in principle that is possible. Our main interest here is that from the solution of the inverse-albedo problem we have presented in the earlier section, the values of f_ℓ are readily obtained from Eq. (2.21). Then, it is quite clear that all quantities in Case's model which involve f_ℓ's

are explicitly calculable. In particular, the inverse-albedo problem provides a method for constructing Case's eigenfunctions from an experiment. Further, it provides an experimental test of the discretized version of the Gelfand-Levitan [5] equation and all equations given by Case, which involve f_ℓ, become identities.

It would be of pedagogical value to find a relation between the direct-inverse problems which is analogous to Eq. (2.28) and involves Case's eigenfunctions.

We make the ansatz in Eq. (2.28), $I(x,\mu) = e^{-x/\nu}\phi_\nu(\mu)$ and obtain

$$\int_{-1}^{1} d\mu' \; S_0(\mu',\mu)\; \mu'\phi_\nu(\mu') = \nu \int_{-1}^{1} d\mu' \; f(\mu'\to\mu)\; \phi_\nu(\mu') \tag{3.17}$$

for all values of ν. But these are nothing but the expansion coefficients of $S_0(\mu',\mu)$ and $f(\mu'\to\mu)$ in Case's eigenfunction base. More importantly, however, the invariance principles give this particular relation from which one can solve for $f(\mu'\to\mu)$, for a given $S_0(\mu',\mu)$ and vice versa by use of the completeness relation of the eigenfunctions.

It appears to us that Case's eigenfunctions may have a natural interpretation. In other words, could these eigenfunctions correspond to some pure state such that given a pure normal mode of radiation (or neutrons), is there a scattering law which conserves this mode without dispersion? If so then one could construct a medium which will pass only one normal mode and not the others. In a sense that would be a filter.

We close this paper with a remark that the inverse-albedo problem for a finite homogeneous medium provides a very simple method of determining the scattering properties of the medium, which appears to be practical for both laboratory and space experiments.

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